Trigonometry – Solutions

1. $\frac{2\sin x}{\cos x-\sin x\tan x}=$ **A.** $\tan 2x$ **B.** $\cot 2x$ **C.** $\tan x$ **D.** $\cot x$ **E.** $\sec x$ **[2009F, A]**

Sol: $\frac{2\sin x}{\cos x-\sin x\tan x}=\frac{2\sin x}{\cos x-\sin x\sin x/\cos x}=\frac{2\sin x\cos x}{\cos^2 x-\sin^2 x}=\frac{\sin 2x}{\cos 2x}=\tan 2x$

2. If $\sin\theta-\cos\theta=0.2$ and $\sin 2\theta=0.96$, find $\sin^3\theta-\cos^3\theta$. [2009S, 0.296]

Sol: $\sin^3\theta-\cos^3\theta=(\sin\theta-\cos\theta)(\sin^2\theta+\sin\theta\cos\theta+\cos^2\theta)$

$=(\sin\theta-\cos\theta)(1+\sin\theta\cos\theta)=(0.2)(1+\frac{1}{2}\sin 2\theta)\ =(0.2)(1+0.96/2)=0.296$

3. In ΔABC, $AB=5$, $BC=9$, $AC=7$. Find the value of $\frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}$. [2008F, $\frac{1}{8}$]

Sol: $\frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}=\frac{\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}\cos\frac{A-B}{2}}=\frac{\frac{1}{2}(\sin A-\sin B)}{\frac{1}{2}(\sin A-\sin B)}=\frac{\sin A-\sin B}{\sin A+\sin B}=\frac{a-b}{a+b}=\frac{9-7}{9+7}=\frac{1}{8}$.

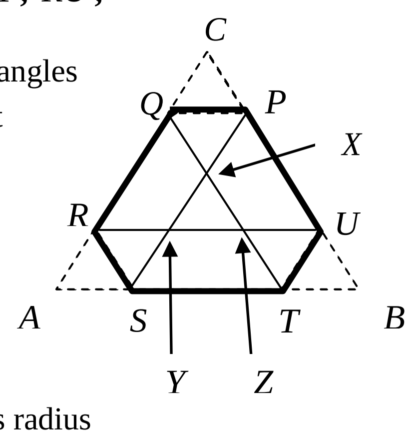
The second equality appeals to the product-to-sum formulas $\sin u\cos v=\frac{1}{2}[\sin(u+v)+\sin(u-v)]$, and $\cos u\sin v=\frac{1}{2}[\sin(u+v)-\sin(u-v)]$. The fourth equality is based on the Law of Sines, $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$, which, simply put, says that the proportion $\sin A:\sin B:\sin C$ is the same as $a:b:c$.

4. In ΔABC, $AB=AC$ and in ΔDEF, $DE=DF$. If AB is twice DE and $\angle D$ is twice $\angle A$, then the ratio of the area of ΔABC to the area of ΔDEF is:

A. $\tan A$ **B.** $2\sec A$ **C.** $\csc 2A$ **D.** $\sec A\tan A$ **E.** $\cot 2A$ **[2008F, B]**

Sol: The area of ΔDEF is $\frac{1}{2}(\overline{DE})^2\sin D=\frac{1}{2}(\frac{1}{2}\cdot\overline{AB})^2\sin(2A)=\frac{1}{8}(\overline{AB})^2 2\sin A\cos A$ $=\frac{1}{2}\cos A[\frac{1}{2}(\overline{AB})^2\sin A]$, which is $\frac{1}{2}\cos A$ times the area of ΔABC. Thus the area of ΔABC is $1/(\frac{1}{2}\cos A)=2\sec A$ times the area of ΔDEF.

5. In hexagon $PQRSTU$, all interior angles $=120°$. If $PQ=RS=TU=50$, and $QR=ST=UP=100$, find the area of the triangle bounded by QT, RU, and PS to the nearest tenth. [2008F, 1082.5]



Sol: The accompanying picture illustrates the situation at hand. All angles are either $60°$ or $120°$. $\overline{RU}=\overline{CR}=\overline{CQ}+\overline{QR}=50+100=150$. But $\overline{RY}=\overline{RS}=50$, and likewise $\overline{ZU}=50$, so $\overline{YZ}=50$. The area of the equilateral triangle ΔXYZ is thus $\frac{1}{2}(50)^2\sin(60°)=\frac{1}{2}(50)^2\frac{\sqrt{3}}{2}=625\sqrt{3}\approx 1082.5$.

6. In ΔABC, $AB=AC=25$, $BC=14$. The perpendicular distances from a point P in the interior of ΔABC to each of the three sides are equal. Find this distance. [2008S, $\frac{21}{4}$]

Sol: P is the center of the inscribed circle of ΔABC and we want its radius r. The area of ΔABC is the sum of the areas of ΔABP, ΔBCP, ΔCAP, i.e. $\frac{1}{2}\overline{AB}\cdot r+\frac{1}{2}\overline{BC}\cdot r+\frac{1}{2}\overline{CA}\cdot r=\frac{1}{2}(25+25+14)r=32r$. On the other hand, Heron's

Formula $\sqrt{s(s-a)(s-b)(s-c)}$, with $s=\frac{1}{2}(a+b+c)=\frac{1}{2}(25+25+14)=32$, gives the area as $\sqrt{32(32-25)(32-25)(32-14)}=168$. So $32r=168$, $r=\frac{21}{4}$.

7. The graph of the function $f(x)=x+\sin kx$ ($|k|\le 1$) intersects the graph of the function $f^{-1}(x)$ at $(4,a)$, $(12,b)$, and $(-8,c)$. Find the value of $a+b+c$. [2007S, 8]

Sol: We claim that the graph of this f and that of its inverse f^{-1} can only meet at points on the line $y=x$. Thus $a=4$, $b=12$, $c=-8$. The answer follows. To prove the claim, let (x, y) be on both the graphs of f and f^{-1}. Then $y=x+\sin kx$, and $x=y+\sin ky$. Take the difference between the two equation and rewrite it to $-2(x-y)=\sin kx-\sin ky$, so $|2(x-y)|=|\sin kx-\sin ky|\le|kx-ky|$ $=|k||x-y|\le|x-y|$, thus $2|x-y|\le|x-y|$, and so $x=y$.

8. If $\cos(\arctan(x))=x$ (x in radians), then x^2 can be expressed in the form $\frac{a+\sqrt{b}}{2}$. Find $a+b$. [2007S, 4]

Sol: $x^2=\cos^2(\arctan x)=\frac{1}{\sec^2(\arctan x)}=\frac{1}{1+\tan^2(\arctan x)}=\frac{1}{1+x^2}$, thus $x^2(x^2+1)=1$, i.e. $(x^2)^2+x^2-1=0$. The quadratic formula gives $x^2=\frac{-1+\sqrt{5}}{2}$.

9. The sum of the solutions of $\arctan\frac{1}{x}+\arctan\frac{1}{x+2}=\arctan\frac{4}{x+4}$ is

A. negative B. even C. 1 D. greater than 5 E. prime [2007S, E]

Sol: Denote $\alpha=\arctan\frac{1}{x}$, $\beta=\arctan\frac{1}{x+2}$, then $\tan\alpha=\frac{1}{x}$, $\tan\beta=\frac{1}{x+2}$, therefore $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{1/x+1/(x+2)}{1-(1/x)(1/(x+2))}=\frac{2x+2}{x^2+2x-1}$. It follows that $\frac{2x+2}{x^2+2x-1}=\frac{4}{x+4}$, which implies $4(x^2+2x-1)=(2x+2)(x+4)$, i.e. $x^2-x-6=0$. Solve to get $x=3,-2$. But -2 doesn't work. So $x=3$. Thus the answer.

10. Let $s(x)=\sin(\pi x)$ and $S(x)=[s(x)]^2$. Find $s(s(1/6))+S(S(1/3))$. [2006F, 3/2]

Sol: Straightforward.

11. In pentagon $AMTYC$, $AC=MT=10$, $YT=CY=20$, $\angle A=\angle M=135°$, and $\angle Y=150°$. Find the area of the pentagon to the nearest square unit. [2006F, 323]

Sol: The pentagon is symmetric. Thus $\angle YTM$ and $\angle YCA$ both equals $\frac{1}{2}(3\cdot 180°-2\cdot 135°-150°)=60°$. With $MT=10$ and $YT=20$, this makes ΔYTM a $30°$- $60°$-$90°$ special triangle, with $MY=10\sqrt{3}$, and an area of $\frac{1}{2}(10)(10\sqrt{3})=50\sqrt{3}$. Likewise ΔYCA has an area of $50\sqrt{3}$. Since $\angle YMT=90°$ and $\angle AMT=135°$, it follows that $\angle AMY=45°$. Likewise $\angle MAY=45°$, so

ΔAMY is a $45°$- $45°$-$90°$ right triangle. As $MY = 10\sqrt{3}$, the area of ΔAMY is $\frac{1}{2}(10\sqrt{3})(10\sqrt{3}) = 150$. The area of the pentagon is thus $50\sqrt{3} + 50\sqrt{3} + 150 \approx 323$.

12. If $f(x) = \cos \pi x$ and $g(x) = 2x$, find $f(g(1)) - g(f(1))$. [2006S, 3]

Sol: Straightforward.

13. If $ABCD$, $DCEF$, $FEGH$ are squares with A, B, C, D, E, F, G and H all disjoint points, find $m\angle GAH + m\angle GDH + m\angle GFH$ to the nearest tenth of a degree. [2006S, $90°$]

Sol: Denote the three angles being summed by α, β, γ, then $\tan\alpha = \frac{1}{3}$, $\tan\beta = \frac{1}{2}$, and $\tan\gamma = 1$, i.e. $\gamma = 45°$. Thus $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\cdot\frac{1}{2}} = 1$, therefore $\alpha + \beta = 45°$, and so $\alpha + \beta + \gamma = 90°$.

14. In convex pentagon $AMTYC$, $\overline{CY} \perp \overline{YT}$, $\overline{MT} \perp \overline{YT}$, $CY = YT = 63$, $MT = 79$, $AM = 39$, and $AC = 52$. Find the area of the pentagon. [2006S, 5487]

Sol: Draw a line segment from C perpendicular to $\overline{MT}$, reaching $\overline{MT}$ at D, then ΔCDM is a right triangle. $DM = 79 - 63 = 16$, $CD = 63$. Use the Pythagorean Theorem to get $CM = 65$. Then $AM : AC : CM = 39 : 52 : 65 = 3 : 4 : 5$, thus $\angle MAC = 90°$, with the area of $\Delta MAC = (AM)(AC)/2 = (39)(52)(2) = 1014$. The trapezoid $CYTM$ has an area of $\frac{1}{2}(63 + 79)(63) = 4473$. We get $1014 + 4473 = 5487$.

15. If α is the acute angle formed by the lines with equations $y = 2x - 5$ and $y = 1 - 3x$, find $\tan\alpha$. [2006S, 1]

Sol: The two lines have slopes 2 and -3, respectively. So they make acute angles β and γ, respectively, with the positive x-axis, such that $\tan\beta = 2$, $\tan\gamma = 3$. Thus $\tan(\beta+\gamma) = \dfrac{\tan\beta + \tan\gamma}{1 - \tan\beta\tan\gamma} = \dfrac{2+3}{1-2\cdot 3} = -1$. So $\beta + \gamma = 135°$, thus $\alpha = 45°$.

16. In the quadrilateral $PQRS$, $PQ = 1$, $QR = RS = \sqrt{2}$, $PS = \sqrt{3}$, and $QS = 2$. If T is the point of intersection of the diagonals, find the measure in degrees of angle RTS. [2006S, 75]

Sol: ΔSQP is a $30°$- $60°$-$90°$ special right triangle, and ΔSQR is a $45°$- $45°$-$90°$ special right triangle. It follows that P, Q, R, S fall on a circle, with $\overline{QS}$ being a diameter. The inscribed angle $\angle PRQ$ equals in measure the inscribed angle $\angle PSQ = 30°$. Thus $\angle RTS = \angle RQT + \angle TRQ = 45° + 30° = 75°$.

17. ΔSML has sides of length 6, 7, 8. Find the exact value of $(\cos S + \cos M + \cos L)$. [2005F, $\frac{47}{32}$]

Sol: Use Cosine Law, $s^2 = m^2 + \ell^2 - 2m\ell\cos S$ to get $\cos S = \dfrac{m^2 + \ell^2 - s^2}{2m\ell}$ $= \dfrac{7^2 + 8^2 - 6^2}{2\cdot 7\cdot 8}$. Likewise compute $\cos M$ and $\cos L$ before summing the three.

18. Find the sum of all solutions of $\cos x = \cot x \cos x$ for which $0 \le x \le 2\pi$. [2005F, 3.5π]

Sol: $\cos x = 0$ or $\cot x = 1$. Thus $x = \frac{1}{2}\pi$, $\frac{3}{2}\pi$, $\frac{1}{4}\pi$, $\frac{5}{4}\pi$. The answer follows.

19. A triangle has vertices $A(0,0)$, $B(3,0)$, and $C(3,4)$. If the triangle is rotated counterclockwise around the origin until C lies on the positive y-axis, find the area of the intersection of the region bounded by the original triangle and the region bounded by the rotated triangle. [2005F, $\frac{21}{16}$]

Sol: Let A', B', C' be the points that A, B, C go after the rotation. In particular, $A' = A$ and C' is at $(0,5)$. $\overline{AC}$ is on the line $y = \frac{4}{3}x$, while $\overline{C'B'}$ is on the line $y = -\frac{4}{3}x + 5$. Thus $\overline{AC}$ and $\overline{C'B'}$ meets at a point D whose x-coordinate satisfies $\frac{4}{3}x = -\frac{4}{3}x + 5$, i.e. $\frac{15}{8}$. The union of right triangular regions $\triangle ABC$ and $\triangle A'B'C'$ is the same as the union of abutting regions $\triangle ABC$ and $\triangle ADC'$, and thus carries an area of $\frac{1}{2}(3)(4) + \frac{1}{2}(5)(\frac{15}{8}) = \frac{171}{16}$, which should equal the sum of the area of $\triangle ABC$ plus that of $\triangle A'B'C'$ minus the overlap area. Thus the overlap area is $\frac{1}{2}(3)(4) + \frac{1}{2}(3)(4) - \frac{171}{16} = \frac{21}{16}$.

20. If $0 < t < \pi/2$, $0 < z < 1$, and $\cos t = \dfrac{1-z^2}{1+z^2}$, how many of the following are true?

$z = \sqrt{\dfrac{1-\cos t}{1+\cos t}}$ **;** $\sin t = \dfrac{2z}{1+z^2}$ **;** $\tan t = \dfrac{2z}{1-z^2}$ **;** $z = \tan\dfrac{t}{2}$ **[2005F, 4]**

Sol: This has to do with a well-known change of variable in calculus, attributed to Karl Weierstrass. One way to manage the situation is to draw a right triangle with one angle being t, the opposite side being $1 - z^2$, and the hypotenuse being $1 + z^2$. The adjacent side can be computed using the Pythagorean Theorem: $\sqrt{(1+z^2)^2 - (1-z^2)^2} = \sqrt{4z^2} = 2z$, then the second and the third formula follow. The first comes from solving $\cos t = \dfrac{1-z^2}{1+z^2}$ for z^2 and then taking square root. The fourth comes from the first by a half-angle formula for tan.