Numbers (Part I) -- Solutions

1. **The equation $a^3+b^3+c^3=2008$ has a solution in which a, b, c are distinct even positive integers. Find $a+b+c$. [2008S, 22]**
   Sol: Let $a=2x$, $b=2y$, $c=2z$. Then $x^3+y^3+z^3=251$. Since $1^3=1$, $2^3=8$, $3^3=27$, $4^3=64$, $5^3=125$, $6^3=216$, $7^3=343$, we see that x, y, z are at most 6. Moreover, since $3\cdot 64=192<251$, the largest of x, y, z must be 5 or 6. Say, $x=6$, then $y^3+z^3=251-216=35$, which is $27+8=3^3+2^3$. So $x+y+z=6+3+2=11$, i.e. $a+b+c=22$. It is not hard to see that this choice of x, y, z (and thus a, b, c) is the only possibility up to reordering.
2. **Each bag to be loaded onto a plane weighs either 12, 18, or 22 lb. If the plane is carrying exactly 1000 lb of luggage, what is the largest number of bags it could be carrying? [2008S, 82]**
   Sol: We want to find whole numbers x, y, z to maximize $x+y+z$ while maintaining $6x+9y+11z=500$ (same as $12x+18y+22z=1000$). One should use smaller bags whenever possible. More precisely, bringing down z by 3 would reduce $6x+9y+11z$, by 33, which can then be compensated for by increasing x by 1 and y by 3 (because $6\cdot 1+9\cdot 3=33$), with $x+y+z$ increased. So when $x+y+z$ is maximized, z must be 0, 1, or 2. Likewise, bringing down y by 2 while increasing x by 3 has the effect of keeping $6x+9y+11z=500$ while increasing $x+y+z$. Therefore, when $x+y+z$ is maximized, y must be 0 or 1. Since in $6x+9y+11z=500$, $6x$ and 500 are even, so $9y+11z$ has to be even. This implies that (y,z) has to be $(0,0)$, $(0,2)$, $(1,1)$. Among them, only the last one would make $500-9y-11z$ divisible by 6. This results in $x=80$, $y=1$, $z=1$, so $x+y+z=82$.
3. **Call a positive integer *biprime* if it is the product of exactly two distinct primes (thus 6 and 15 are biprime, but 9 and 12 are not). If N is the smallest number such that N, $N+1$, $N+2$ are all biprime, find the largest prime factor of $N(N+1)(N+2)$. [2008S, 17]**
   Sol: Observe that a biprime cannot be a multiple of 4. Hence N has to be congruent to 1 modulo 4 (i.e. of the form $4k+1$), for otherwise at least one of the three numbers would be a multiple of 4. It follows that $N+1$ is even. But $N+1$ is coprime, so $N+1=2p$, with $p\neq 2$ a prime, and thus $N+1=$ 6, 10, 14, 22, 26, 34, …. We quickly rule out $N+1=6$, through 26, as either N or $N+2$ would fail to be a coprime. Thus N, $N+1$, $N+2$ are 33, 34, 35, with $N(N+1)(N+2)=3\cdot 11\cdot 2\cdot 17\cdot 7\cdot 5$. The answer follows.
4. **Let r, s, t be nonnegative integers. For how many such triples (r,s,t) satisfying the system $\begin{cases} rs+t = 24 \\ r+st = 24 \end{cases}$ is it true that $r+s+t=25$? [2008S, 26]**
   Sol: Take the difference of the two equations in the system to get $r(s-1)+t(1-s)=0$, i.e. $(s-1)(r-t)=0$. <u>Case 1</u>: $s=1$. Thus the system says $r+t=24$ and $r+s+t=25$ also becomes $r+t=24$. There are 25 possible triples

(r, s, t) of this kind. Case 2: $s \neq 1$, thus $r = t$. The system says $t(s+1) = 24$, while $r+s+t = 25$ says $2t+s = 25$. So we have $\begin{cases} t(s+1) & = & 24 \\ 2t+s & = & 25 \end{cases}$, which means $\begin{cases} (2t)(s+1) & = & 48 \\ (2t)+(s+1) & = & 26 \end{cases}$, therefore $2t$ and $s+1$ form the two roots of the quadratic equation $x^2 - 26x + 48 = 0$, i.e. $(x-2)(x-24) = 0$. Since $s \neq 1$, it follows that $s+1 = 24$, $2t = 2$, so $(r, s, t) = (1, 23, 1)$. Cases 1 and 2 combined give 26 triples.

5. **The digits 1 to 9 can be separated into 3 disjoint sets of 3 digits each so that the digits in each set can be arranged to form a 3-digit perfect square. Find the last two digits of the sum of these three perfect squares.**
A. 26 B. 29 C. 34 D. 46 E. 74 [2008S, E]
Sol: Approach the problem by brute force. First, list all perfect squares formed by three distinct non-zero digits by squaring 10, 11, 12, etc and ruling out those with repeated digits. The resulting list is 169, 196, 256, 289, 324, 361, 529, 576, 625, 729, 784, 841, 961. Among them only 324 and 361 contain the digit 3, thus one of them has to be included. If it's 324, then the other two perfect squares must be from among 169, 196, 576, 961, in order to avoid reusing 3, 2, or 4. For the digit 5 to appear, we are forced to include 576, with the remaining digits 1, 8, 9 unable to form a perfect square, thus 324 doesn't work. Try 361, with the other two perfect squares thus coming from among 289, 529, 729, 784. To accommodate the digit 5, we include 529, with the remaining perfect square thus being 784. We have $361 + 529 + 784 = 1674$.

6. **The sequence $\{a_n\}$ is defined by $a_0 = a_1 = a_2 = 1$, and $a_{n-3}a_n - a_{n-2}a_{n-1} = (n-3)!$ for $n \geq 3$. If 5^k is the largest power of 5 that is a factor of $a_{100}a_{101}$, find k.**
[2008S, 24]
Sol: Divide $a_{n-3}a_n - a_{n-2}a_{n-1} = (n-3)!$ by $a_{n-2}a_{n-3}$ on both sides to get $\frac{a_n}{a_{n-2}} - \frac{a_{n-1}}{a_{n-3}} = \frac{(n-3)!}{a_{n-2}a_{n-3}}$. Try the first few iterations to see if there is any pattern:

$\frac{a_3}{a_1} - \frac{a_2}{a_0} = \frac{0!}{a_1a_0}$, i.e. $\frac{a_3}{a_1} - \frac{1}{1} = \frac{0!}{1 \cdot 1} = 1$, thus $\frac{a_3}{a_1} = 2$ and $a_3 = 2a_1 = 2$

$\frac{a_4}{a_2} - \frac{a_3}{a_1} = \frac{1!}{a_2a_1}$, i.e. $\frac{a_4}{a_2} - 2 = \frac{1!}{1 \cdot 1} = 1$, thus $\frac{a_4}{a_2} = 3$ and $a_4 = 3a_2 = 3 \cdot 1 = 3$

$\frac{a_5}{a_3} - \frac{a_4}{a_2} = \frac{2!}{a_3a_2}$, i.e. $\frac{a_5}{a_3} - 3 = \frac{2!}{2 \cdot 1} = 1$, thus $\frac{a_5}{a_3} = 4$ and $a_5 = 4a_3 = 4 \cdot 2 = 8$

$\frac{a_6}{a_4} - \frac{a_5}{a_3} = \frac{3!}{a_4a_3}$, i.e. $\frac{a_6}{a_4} - 4 = \frac{3!}{3 \cdot 2} = 1$, thus $\frac{a_6}{a_4} = 5$ and $a_6 = 5a_4 = 5 \cdot 3 = 15$

We begin to suspect that $\frac{a_n}{a_{n-2}} = n-1$ in general, which then leads to $a_n = (n-1)a_{n-2} = (n-1)(n-3)a_{n-4} = \ldots$. For example, $a_6 = 5 \cdot 3 \cdot 1$, $a_7 = 6 \cdot 4 \cdot 2$, $a_8 = 7 \cdot 5 \cdot 3 \cdot 1$, etc. If such pattern persists up to a_{n-1}, then $\frac{a_n}{a_{n-2}} - \frac{a_{n-1}}{a_{n-3}} = \frac{(n-3)!}{a_{n-2}a_{n-3}}$ becomes $\frac{a_n}{a_{n-2}} - (n-2) = \frac{(n-3)!}{(n-3)!} = 1$, and thus $\frac{a_n}{a_{n-2}} = n-1$, and so the induction can be continued to uphold the pattern. With the pattern now established, it follows that $a_{100}a_{101} = a_{101}a_{100} = 100!$ We are left with finding how many factors of 5 the number 100! contains. Among the one hundred numbers 1 through 100, twenty numbers are multiples of 5, i.e. 5, 10, 15, 20, …, 95, 100. Among these, 25, 50, 75, 100 have the factor 5 appearing two times. So, altogether 100! has 20+4=24 factors of 5. The answer follows.

7. **Trina has two dozen coins, all dimes and nickels, worth between $1.72 and $2.11. What is the least number of dimes she could have? [2007F, 11]**
Sol: Let there be x dimes and y nickels. Thus $x+y=24$, while $172<10x+5y<211$. Since $10x+5y$ is divisible by 5, so $175\le 10x+5y\le 210$, i.e. $35\le 2x+y\le 42$, i.e. $35\le x+(x+y)\le 42$. As $x+y=24$, we get $11\le x\le 18$. So the least possible value for x is 11 (and thus $y=13$.)

8. **Replace each letter of AMATYC with a digit 0 through 9 to form a six-digit number (identical letters are replaced by identical digits, different letters are replaced by different digits). If the resulting number is the largest such number which is a perfect square, find the sum of its digits (that is, $A+M+A+T+Y+C$)**
A. 32 B. 33 C. 34 D. 35 E. 36 [2007F, E]
Sol: We first try 989XXX. Such numbers do not exceed 989999. Use a calculator to get $\sqrt{989999}\approx 994.987$, so if 989XXX is a perfect square, it has to be at most $994^2 = 988036$, which is already too small to be of the form 989XXX. This rules out 989XXX. The same approach can be used to rule out 979XXX, 969XXX, …, 939XXX. For 929XXX, we do $\sqrt{929999}\approx 964.365$, and thus we compute $964^2 = 929296$, which is of the form 929XXX, but doesn't satisfy the required form AMATYC, while $963^2 = 927369$ is already too small to be of the form 929XXX. This rules out 929XXX. 919XXX can also be ruled out this way. 909XXX is also ruled, in a manner similar to 989XXX. We conclude that A cannot be 9. We then try 898XXX, using the same method, and immediately hit $948^2 = 898704$. The answer follows.

9. **Add any integer N to the square of $2N$ to produce an integer M. For how many values of N is M prime? [2007F, 2]**
Sol: $N+(2N)^2 = N+4N^2 = N(4N+1)$. Since this is a prime, at least one of the following conditions is met: (1) $N=1$ (2) $N=-1$ (3) $4N+1=1$ (4) $4N+1=-1$. Of these, only (1) and (2) make sense. The answer follows.

10. **When certain proper fractions in simplest terms are added, the result is in simplest terms: $\frac{2}{15}+\frac{1}{21}=\frac{19}{105}$; in other cases, the result is not in simplest terms: $\frac{2}{15}+\frac{5}{21}=\frac{39}{105}=\frac{13}{35}$. Assume that $\frac{m}{15}$ and $\frac{n}{21}$ are positive proper fractions in simplest terms. For how many such fractions is $\frac{m}{15}+\frac{n}{21}$ not in simplest terms? [2007F, 48]**
Sol: $\frac{m}{15}+\frac{n}{21}=\frac{7m+5n}{105}$. Since $\frac{m}{15}$ and $\frac{n}{21}$ are positive proper fractions in simplest terms, we have $m=$ 1, 2, 4, 7, 8, 11, 13, 14 and $n=$1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20. Since $\frac{7m+5n}{105}$ is not in simplest terms, $7m+5n$ is divisible by 5, 7, or 3. But it is not divisible by 5 because m is not divisible by 5. Likewise $7m+5n$ is not divisible by

7 either. It follows that $7m+5n$ is divisible by 3. But $7m+5n=6(m+n)+(m-n)$, so this is the same as saying $m-n$ is divisible by 3, i.e. $m \equiv n$ mod 3. Among the eight possible values for m, 1, 4, 7, 13 are congruent to 1 mod 3, while the remaining 2, 8, 11, 14 are congruent to 2 mod 3. For the twelve possible values for n, 1, 4, 10, 13, 16, 19 are congruent to 1 mod 3, while the remaining 2, 5, 8, 11, 17, 20 are congruent to 2 mod 3. If m is congruent to 1 mod 3, then so is n, with $4\times 6=24$ combinations. Likewise, if m is congruent to 2 mod 3, then so is n, with $4\times 6=24$ combinations. Together we have therefore 48 scenarios.

11. Let r, s, and t be nonnegative integers. How many such triples (r, s, t) satisfy the system $\begin{cases} rs+t=14 \\ r+st=13 \end{cases}$? [2007F, 2]

Sol: Subtract the second equation from the first equation to get $r(s-1)+t(1-s)=1$, i.e. $(s-1)(r-t)=1$. Two cases: Case 1: $\begin{cases} s-1=1 \\ r-t=1 \end{cases}$, so $s=2$, and the original system becomes $\begin{cases} 2r+t=14 \\ r+2t=13 \end{cases}$, which gives $r=5$, $t=4$. Case 2: $\begin{cases} s-1=-1 \\ r-t=-1 \end{cases}$, so $s=0$, and the original system becomes $\begin{cases} t=14 \\ r=13 \end{cases}$. We conclude that there are two such triples.

12. If AM/AT = .YC, where each letter represents a different digit, AM/AT is in simplest terms, and A $\neq$ 0, then AT =? [2007S, 25]

Sol: $\frac{AM}{AT}=\frac{YC}{100}$. So AT is a factor of 100. There are nine factors of 100, among them the following have two digits: 10, 20, 25, 50. But $\frac{AM}{AT}$ has to be a proper fraction, therefore AT cannot be 10, 20, nor 50. The answer follows.

13. Two adjacent faces of a rectangular box have areas 36 and 63. If all three dimensions are positive integers, find the ratio of the largest possible volume of the box to the smallest possible volume. [2006F, 9]

Sol: This means there are positive integers a, b, c with $ab=36$ and $bc=63$, so b is a common factor of 36 and 63. Since the GCF of 36 and 63 is 9, this means b is a factor of 9, and so $b=1$, 3, 9. The volume is $abc=\frac{(ab)(bc)}{b}=\frac{(36)(63)}{b}$. It follows that the largest and smallest possible values are $\frac{(36)(63)}{1}$ and $\frac{(36)(63)}{9}$. The answer follows.

14. In the expression (AM)(AT)(YC), each different letter is replaced by a different digit 0 to 9 to form three two-digit numbers. If the product is to be as large as possible, what are the last two digits of the product?
A. 20 B. 40 C. 50 D. 60 E. 90 [2006F, B]

Sol: Try $(95)(96)(87)=793440$. We will convince ourselves that this is the largest possible. First of all, at least one of the three numbers have to be at least 90, for otherwise the product would be at most $(89)(89)(89)=704969<793440$. But Y cannot be 9, for otherwise the product would be at most $(89)(89)(98)=776258<793440$. It follows that A is 9. Then Y has to be 8, for otherwise the product would be at most $(98)(98)(79)=758716<793440$. We are left with only (9M)(9T)(8C), and we only have to compare the three products:

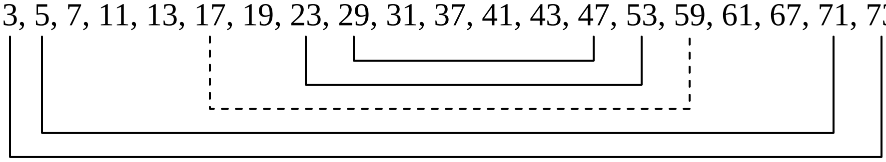
(95)(96)(87), (95)(97)(86), and (96)(97)(85), of which $(95)(96)(87) = 793440$ is the largest.

15. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 5 & -10 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, find the smallest possible value of $a+b+c+d$, if a, b, c, and d are all positive integers. [2006F, 16]

Sol: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 5 & -10 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} 5a-3b & -10a+6b \\ 5c-3d & -10c+6d \end{bmatrix} = \begin{bmatrix} 5a-3b & -2(5a-3b) \\ 5c-3d & -2(5c-3d) \end{bmatrix}$. So the condition that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 5 & -10 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is equivalent to $\begin{cases} 5a-3b=0 \\ 5c-3d=0 \end{cases}$. But $5a-3b=0$ means $a:b=3:5$. So there is a positive integer u such that $a=3u$, $b=5u$. Likewise, $c=3v$, $d=5v$ for some positive integer v. So $a+b+c+d=8u+8v=8(u+v)$. The answer follows.

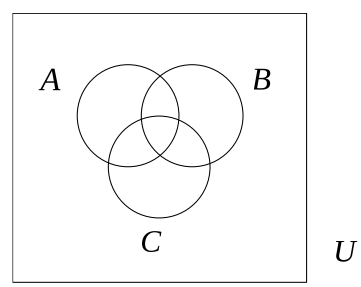
16. The year 2006 is the product of exactly three distinct primes p, q, and r. How many other years are also the product of three distinct primes with sum equal to $p+q+r$? [2006F, 4]

Sol: $2006 = 2\times17\times59$, with $2+17+59=78$. Thus we want whole numbers that are the product of three distinct primes whose sum is 78. Note that 78 is even, so one of the three primes must be 2, with the sum of the other two being 76. List all primes smaller than 80 as follows, from which we identify pairs that sum to 76:
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79



The answer follows.

17. How many positive integers less than 1000 are relatively prime to 105? Two integers are relatively prime if their greatest common divisor is 1. [2006F, 457]



Sol: Let U be the set of all positive integers less than 1000. Let A, B, C be the subsets of U containing those that are divisible by 3, 5, 7, respectively. Then the problem aims to determine $\#(U-(A\cup B\cup C))$. This equals $\#U-[\#A+\#B+\#C-\#(A\cap B)-\#(B\cap C)-\#(C\cap A)+\#(A\cap B\cap C)]$. Note that, for example, $A\cap B$ is the subset of U formed by those that are divisible by $3\times5=15$, thus $\#(A\cap B)=66$ since $999=15\times66+9$. Likewise, e.g. $A\cap B\cap C$ is the subset formed by those that are divisible by $3\times5\times7=105$. Thus we get $999-[333+199+142-66-28-47+9]=457$

18. How many 4-digit numbers whose digits are all odd are multiples of 11? [2006F, 85]

Sol: Let ABCD be such a 4-digit number, with each of A, B, C, D being any of 1, 3, 5, 7, 9. Divisibility of 11 is equivalent to the condition that the difference between the sum A+C and the sum B+D is divisible by 11. But either sum is at least $1+1=2$ and at most $9+9=18$, so the difference must be –11, 0, or 11. Since both A+C and

B+D are even, we rule out –11 and 11, thus A+C=B+D. List all possible ordered pair from 1, 3, 5, 7, 9, grouped according to the sum:
Sum = 2: 1+1
Sum = 4: 1+3, 3+1
Sum = 6: 1+5, 5+1, 3+3
Sum = 8: 1+7, 7+1, 3+5, 5+3
Sum = 10: 1+9, 9+1, 3+7, 7+3, 5+5
Sum = 12: 3+9, 9+3, 5+7, 7+5
Sum = 14: 5+9, 9+5, 7+7
Sum = 16: 7+9, 9+7
Sum = 18: 9+9
Both A+C and B+D have to be from the same group. There are thus $1^2+2^2+3^2+4^2+5^2+4^2+3^2+2^2+1^2=85$ possibilities.

19. Find the tens digit of 3^{2007}. [2006F, 8]

Sol 1: $3^{2007}=3(3^{2006})=3(9^{1003})=3\cdot(10-1)^{1003}$. By the binomial expansion, we see

$$(10-1)^{1003}$$
$$=10^{1003}+1003\cdot10^{1002}(-1)^1+\binom{1003}{2}\cdot10^{1001}(-1)^2+\cdots+\binom{1003}{1001}\cdot10^2(-1)^{1001}+1003\cdot10^1(-1)^{1002}+(-1)^{1003}$$

Thus $(10-1)^{1003}\equiv1003\cdot10^1(-1)^{1002}+(-1)^{1003}\equiv10030-1\equiv10029$ mod 100. So $3^{2007}\equiv3\cdot(10-1)^{1003}\equiv(3)(10029)\equiv30087\equiv87$ mod 100. The answer follows.

Sol 2: Working modulo 100, we have $3^1\equiv3$, $3^2\equiv9$, $3^3\equiv27$, $3^4\equiv81$, $3^5\equiv43$, $3^6\equiv29$, $3^7\equiv87$, $3^8\equiv61$, $3^9\equiv83$, $3^{10}\equiv49$, $3^{11}\equiv47$, $3^{12}\equiv41$, $3^{13}\equiv23$, $3^{14}\equiv69$, $3^{15}\equiv7$, $3^{16}\equiv21$, $3^{17}\equiv63$, $3^{18}\equiv89$, $3^{19}\equiv67$, $3^{20}\equiv1$, $3^{21}\equiv3$. We thus notice that every 20 steps the thing repeats. It follows that $3^{2007}\equiv3^7$ mod 100, i.e. $3^{2007}\equiv87$ mod 100. The answer follows.

20. In the sequence a_1, a_2, a_3, …, $a_1=1$, $a_2=2$, $a_3=5$, and for all $n\geq3$, $a_{n-1}a_{n-2}=2a_na_{n-2}-2a_{n-1}a_{n-1}$. Find a_{2006}/a_{2005}. [2006F, 1004]

Sol: Divide $a_{n-1}a_{n-2}=2a_na_{n-2}-2a_{n-1}a_{n-1}$ on both sides by $a_{n-1}a_{n-2}$ to get $1=\frac{2a_n}{a_{n-1}}-\frac{2a_{n-1}}{a_{n-2}}$, i.e. $\frac{a_n}{a_{n-1}}-\frac{a_{n-1}}{a_{n-2}}=\frac{1}{2}$. This shows that $\frac{a_2}{a_1}$, $\frac{a_3}{a_2}$, $\frac{a_4}{a_3}$, … form an arithmetic sequence with common difference $\frac{1}{2}$. The first term is $\frac{a_2}{a_1}=\frac{2}{1}=2$, so the term $\frac{a_{2006}}{a_{2005}}$ is $2+(2005-1)\cdot\frac{1}{2}=1004$.