Combinatorics/Probability (Part I) -- Solutions

1. **A fair coin is labeled A on one side and M on the other; a fair die has two sides labeled T, two labeled Y, and two labeled C. The coin and die are each tossed three times. Find the probability that the six letters can be arranged to spell AMATYC. [2008S, $\frac{1}{12}$]**

   Sol: Among the 2^3 equally likely outcomes of the three tosses of the coin, 3 have exactly two A's turned up. Among the 3^3 equally likely outcomes of the three tosses of the die, $3! = 6$ have exactly one T, one Y, and one C. So the answer is $\frac{(3)(3!)}{(2^3)(3^3)} = \frac{1}{12}$

2. **The letters AMATYC are written in order, one letter to a square of graph paper, to fill 100 squares. If three squares are chosen at random without replacement, find the probability to the nearest 1/10 of a percent of getting three A's. [2008S, 3.7%]**

   Sol: The complete sequence AMATYC occurs 16 times, taking up $16 \times 6 = 96$ squares. The remaining four squares are AMAT. Thus there are $2 \times 16 + 2 = 34$ A's out of 100 squares. The answer is thus $\frac{34}{100} \cdot \frac{33}{99} \cdot \frac{32}{98} \approx 0.0370068 \approx 3.7\%$

3. **A student committee must consist of two seniors and three juniors. Five seniors are able to serve on the committee. What is the least number of junior volunteers needed if the selectors want at least 600 different possible ways to pick the committee? [2008S, 9]**

   Sol: Let there be n juniors. Want $\binom{5}{2}\binom{n}{3} \geq 600$, i.e. $\frac{5 \times 4}{1 \times 2} \cdot \frac{n(n-1)(n-2)}{1 \times 2 \times 3} \geq 600$, namely $n(n-1)(n-2) \geq 360$. Starting from $n = 3$, $n(n-1)(n-2)$ goes up as n increases. Direct computation quickly shows that it is $n = 9$ when it first reaches at least 360.

4. **How many different 3-letter strings can be formed from the letters of MATHEMATICS (no letter can be used in a given string more times than it appears in the word)? [2007S, 399]**

   Sol: The answer is $8 \times 7 \times 6 + 3 \times 7 \times 3 = 399$. The number $8 \times 7 \times 6$ counts three-letter strings with no letter repeating. The number $3 \times 7 \times 3$ counts three-letter strings where one letter appears twice. The leading factor 3 is for the 3 choices for the repeating letter (M, A, or T); For each, there are 7 choices for the non-repeating letter; The last factor of 3 counts for the 3 possible slots the non-repeating letter can go.

5. **A positive integer less than 1000 is chosen at random. What is the probability it is a multiple of 3, but a multiple of neither 2 nor 9. [2006S, $\frac{1}{9}$]**

   Sol: If k is a natural number, define $U_k = \{x | 1 \leq x \leq 999, x \text{ divisible by } k\}$. A Venn diagram shows $\#(U_3 \setminus (U_2 \cup U_9)) = \#(U_3) - \#(U_3 \cap U_2) - \#(U_3 \cap U_9) + \#(U_3 \cap U_2 \cap U_9)$. Note that $U_3 \cap U_2 = U_6$, $U_3 \cap U_9 = U_9$, $U_3 \cap U_2 \cap U_9 = U_{18}$ (18 being the LCM of 3, 2, 9). So $\#(U_3 \setminus (U_2 \cup U_9)) = \#(U_3) - \#(U_6) - \#(U_9) + \#(U_{18})$. i.e. $333 - 166 - 111 + 55 = 111$. The answer is $\frac{111}{999} = \frac{1}{9}$.

6. **A palindrome is a word or a number (like RADAR or 1221) which reads the same forwards and backwards. If dates are written in the format MMDDYY, how many dates in the 21st century are palindromes? [2005S, 24]**

Sol: MM can be any of 01, 02, 03, …, 10, 11, 12, with the corresponding YY being 10, 20, 30, …, 01, 11, 21. DD can be 11 or 22. So, we have $12\times 2 = 24$ palindromes.

7. **Mrs. Abbott finds that the number of possible groups of 3 students in her class is exactly five times the number of possible groups of 2 students. How many students are in her class? [2005S, 17]**

   Sol: $\binom{n}{3} = 5\cdot\binom{n}{2}$, i.e. $\frac{n(n-1)(n-2)}{1\times 2\times 3} = \frac{5n(n-1)}{1\times 2}$, so $\frac{n-2}{3} = 5$, thus $n = 17$.

8. **Teams A and B play a series of games; whoever wins two games first wins the series. If Team A has a 70% chance of winning any single game, what is the probability that Team A wins the series? [2004S, 0.784]**

   Sol: The winning team sequence before victory is declared for team A could be any of the following: AA, BAA, ABA, mutually exclusive. The probability for these are, respectively, $(0.7)(0.7)$, $(0.3)(0.7)(0.7)$, $(0.7)(0.3)(0.7)$. Sum them to get 0.784.

9. **A piece has 2 saxophone parts, 3 trumpet parts, and 3 trombone parts. If a band has 2 saxophonists, 3 trumpeters, and 3 trombonists, in how many ways can different parts be assigned to each player? [2007F, 72]**

   Sol: $(2!)(3!)(3!) = (2)(6)(6) = 72$.

10. **Five students enroll in a statistics class. The first test is scored on a percent basis (0% to 100%) rounding each score to the nearest whole number. Four of their scores are 93, 96, 99, and 100. How many possible whole number scores on the fifth student's test will make the median of the five scores equal to the mean of the five scores?**

    A. 0 B. 1 C. 2 D. 3 E. more than 3 [2006F, C]

    Sol: Let m stand for the median of the five scores, and let x stand for the fifth score. We want $5m$ to equal the sum of the five scores. If $x \le 96$, then the median m is 96, so we want $5\times 96 = 93+96+99+100+x$, thus $x = 92$, which is in the range $x \le 96$. If $99 \le x$, then the median is 99, so we want $5\times 99 = 93+96+99+100+x$, so $x = 107$, which is impossible. If $96 < x < 99$, then the median m is x itself, in which case, we would need $5x = 93+96+99+100+x$, so $x = 97$, which is indeed in $96 < x < 99$. To conclude, the fifth score can be 92 or 97.

11. **A basketball player has a constant probability of 80% of making a free throw. Find the probability that her next successfully free throw is the third or fourth one she attempts. [2006F, 0.0384]**

    Sol: $P(FFS) + P(FFFS) = (0.2)(0.2)(0.8) + (0.2)(0.2)(0.2)(0.8) = 0.0384$. (Note that FFS and $FFFS$ are mutually exclusive events.)

12. **If you have eight pairs of socks, each pair a different color, find the probability that if you randomly lose five socks, the remaining socks form exactly four matching pairs (and three unmatched socks). [2006F, $\frac{20}{39}$]**

    Sol: $\dfrac{8\cdot\binom{7}{3}\cdot 2^3}{\binom{16}{5}} = \dfrac{8\cdot\left(\frac{7\times 6\times 5}{1\times 2\times 3}\right)\cdot 8}{\left(\frac{16\times 15\times 14\times 13\times 12}{1\times 2\times 3\times 4\times 5}\right)} = \dfrac{20}{39}$ Here the denominator $\binom{16}{5}$ is the total number of ways to lose 5 socks out of 16. One of the eight pairs will have both socks lost; this accounts for the factor of 8. Three of the remaining 7 pairs have exactly one sock each lost; the factor $\binom{7}{3}$ accounts for which three pairs, whereas 2^3 accounts for the manner in which the three pairs lose one sock each.

13. **A circle contains 25 points chosen so that the arcs between any two adjacent points are equal. Three of these points are chosen at random. Let the probability that the triangle formed is right be R, and the probability that the triangle formed is isosceles be I. Find $|R-I|$. [2005F, $\frac{3}{23}$]**

    Sol: For the triangle to be a right triangle, one side has to be a diameter, which is impossible since 25 is an odd number. Thus $R=0$. Since 25 is not divisible by 3, an isosceles triangle formed cannot be equilateral. Thus the isosceles triangle has a distinguished vertex, which can be any of the 25 points. The remaining 24 points form two groups, 12 consecutive points each. Pick one point from a group, which is paired up with an obvious point from the other group. So the answer is $\frac{25\times 12}{\binom{25}{3}}=\frac{3}{23}$.

14. **Let $A=\{0,1,2,3,4,5,6,7,8,9\}$. How many three-element subsets of A contain at least two consecutive integers? [2004F, 64]**

    Sol 1: For the three elements to be consecutive, there are 8 possibilities. For the first two to be consecutive while the third isn't right next to the second, there are $7+6+5+4+3+2+1=28$ possibilities, where, for example, "7" accounts for cases where the first two are 0 and 1, with the third being anywhere from 3 through 9. Likewise, there are 28 possibilities for which the second and the third are consecutive but the first is not right below the second. The answer is thus $8+28+28=64$.

    Sol 2: [A more sleek approach] There are $\binom{10}{3}=\frac{10\times 9\times 8}{1\times 2\times 3}=120$ ways to form a three-element subset. From these, we will exclude those that do not contain consecutive integers. In such a case, the three elements divide the space into four "compartments", the middle two of which are nonempty. (E.g., for {0, 3, 5}, the first "compartment" [the space to the left of 0] contains nothing, the second, between 0 and 3, contains two elements (1 and 2), the third contains one element (4), and the fourth contains four elements.). The number of subsets of this sort equals the number of ways to put 7 identical beads in four buckets in a row, requiring the middle two buckets to be nonempty. This in turn equals the number of ways to write 7 in the form $k_1+(1+k_2)+(1+k_3)+k_4$, with non-negative integers $k_i\ge 0$. This then is the same as counting the way to partition 5 as a sum of the form $k_1+k_2+k_3+k_4$, with $k_i\ge 0$. This is further equivalent to arranging three 1's and five 0's in a row, because, e.g., the partition $1+0+4+0$ can be encoded as 01100001, with the three 1's dividing the space into four chambers, the first containing one zeros, the second containing none, the third containing four 0's, and the fourth containing none. This last counting task gives $\binom{3+5}{3}=\binom{8}{3}=\frac{8\times 7\times 6}{1\times 2\times 3}=56$. The final answer is $120-56=64$.

15. **A store has four open checkout stands. In how many ways could six customers line up at the checkout stands? [2004F, 60480]**

Sol: First consider the lengths of lines at the four checkout stands. This is the same as asking in how many ways we can arrange 6 zeros and 3 ones in a row. E.g., 001000110 means 2 customers in the first checkout stand, 3 at the second, none at the third, and 1 at the fourth. Here the three 1's are used as boundaries creating four "compartments", each of which accommodates a few 0's. Viewing this problem from this angle, it follows that the pattern of lengths of waiting lines has $\binom{6+3}{3}=\binom{9}{3}=\frac{9\times 8\times 7}{1\times 2\times 3}$ possibilities. For each pattern, we now assign the customers one by one in the 6 waiting spots. There are $6!$ ways to achieve this. Put together, the answer is $\binom{6+3}{3}\cdot 6!=\binom{9}{3}\cdot 6!=\frac{9\times 8\times 7}{1\times 2\times 3}\cdot 1\times 2\times 3\times 4\times 5\times 6=60480$.

16. Consider all arrangements of the letters AMATYC with either the A's together or the A's on the ends. What fraction of all possible such arrangements satisfies these conditions? [2003F, $\frac{2}{5}$]

Sol: As a whole, there are $\frac{6!}{2!}$ ways to arrange the letters AMATYC. Here we first come up with $6!$ by pretending that the two A's were distinguishable. The overcounting is then compensated for by division by $2!$. For the two A's to be together, we imagine that the two A's are bundled together to form an object on equal footing with M, T, Y, C; there are then $5!$ ways to arrange such 5 objects. Finally, if the two A's are on the ends, then we have to arrange M, T, Y, C in the space between the two A's; there are $4!$ ways to achieve this. So the answer is $\frac{5!+4!}{6!/2!}=\frac{2}{5}$.